



PAPER

Informational approach to the quantum symmetrization postulate

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**Abstract**

A remarkable feature of quantum theory is that particles with identical intrinsic properties must be treated as indistinguishable if the theory is to give valid predictions in all cases. In the quantum formalism, indistinguishability is expressed via the symmetrization postulate (Dirac P 1926 *Proc. R. Soc. A* **112** 661, Heisenberg W 1926 *Z. Phys.* **38** 411), which restricts a system of identical particles to the set of symmetric states ('bosons') or the set of antisymmetric states ('fermions'). However, the physical basis and range of validity of the symmetrization postulate has not been established. A well-known topological derivation of the postulate implies that its validity depends on the dimensionality of the space in which the particles move (Laidlaw M and DeWitt C 1971 *Phys. Rev. D* **3** 1375–8, Leinaas J M and Myrheim J 1977 *Il Nuovo Cimento B* **37** 1–23). Here we show that the symmetrization postulate can be derived by strictly adhering to the informational requirement that particles which cannot be experimentally distinguished from one another are not labelled. Our key novel postulate is the *operational indistinguishability postulate*, which posits that the amplitude of a process involving several indistinguishable particles is determined by the amplitudes of all possible transitions of these particles when treated as distinguishable. The symmetrization postulate follows by requiring consistency with the rest of the quantum formalism. The derivation implies that the symmetrization postulate admits no natural variants. In particular, the possibility that identical particles generically exhibit anyonic behavior in two dimensions is excluded.

The treatment of identical particles is one of the most remarkable features of quantum theory. In classical mechanics, identical particles can be treated as distinguishable since one can in principle follow the precise trajectory of each particle over time. However, identical particles in quantum theory must be treated as indistinguishable if the theory is to account for such phenomena as the chemical behaviour of multi-electron atoms.

Formally, the indistinguishability of the particles in a quantum system is expressed via the *symmetrization postulate* [1, 2], which stipulates that a system consisting of a number of identical subsystems must be described by a state that is either symmetric or antisymmetric under any interchange of the state variables. For example, a wavefunction $\psi(x_1, x_2, x_3)$ describing a state of three structureless identical particles must satisfy $\psi(x_1, x_2, x_3) = \pm\psi(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ where π is any transposition, where the choice of sign is fixed for a given system. This postulate is responsible for the classification of all identical particles in nature into bosons and fermions, and is required for the understanding of a vast range of physical phenomena ranging from atomic and molecular structure to the spectrum of blackbody radiation.

However, the physical basis and range of validity of the symmetrization postulate remains uncertain. Shortly after positing the symmetrization postulate, Dirac noted that 'other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles' [5]. In the subsequent years, there have been numerous efforts to determine its physical basis and range of validity [3, 4, 6–14]. However, the results are generally incomplete or inconclusive, and leave open the possibility that identical particles other than fermions and bosons may exist in nature. For example, the well-known topological approach [3, 4] yields the symmetrization postulate in the special case of structureless particles moving in three or more spatial dimensions, but implies the breakdown of the symmetrization postulate in two spatial dimensions with so-

called anyonic behaviour [4, 15]. However, the topological argument is limited to structureless particles, and relies upon abstract assumptions such as the much-scrutinized assumption that point-particles cannot coincide and the conjectured generalization of Feynman's sum rule for multiply-connected spaces [16]. Consequently, although anyons have found significant application [17, 18], for example in the explanation of the fractional Hall effect, their status as *elementary* particles on a par with fermions and bosons remains uncertain.

A significant problem in most of the aforementioned attempts to understand the symmetrization postulate is their labelling of indistinguishable particles. Although labelling of entities that have no physical counterparts is a commonly accepted formal device in physical theories so long as the predictions are invariant under relabelling, the labelling of indistinguishable particles presents special difficulties [19–22], which has led to some approaches which explicitly attempt to avoid labelling of indistinguishable particles (see, for example, [4]). The essential difficulty is that, if one mentally labels indistinguishable particles, and one then considers operations on these particles such as swapping their locations, then one has implicitly assumed that the labels persist through time, which contradicts the initial assumption that the particles are indistinguishable. Furthermore, even if one shows that the predictions are invariant under relabelling, one cannot rule out the possibility that the predictions depend on the fact that one has used labels in the first place. Therefore, to guard against inadvertent inconsistencies or implicit assumptions, it is desirable to refrain from such labelling. Such a point of view is in accord with the informational view of physical theory, with its insistence that theoretical distinctions be underpinned as far as possible by experimentally measurable data, a view that has historically been pivotal in the development of physics, leading, for example, to the abolition of absolute time, a mental construct that has no known experimentally-accessible physical counterpart.

In recent years, there has been growing appreciation that an informational viewpoint provides a powerful vantage point for establishing the physical basis and range of validity of the mathematical formalism of quantum theory [23–26]. Substantial progress has now been made, both in deriving much of the quantum formalism from informational principles [27–34], and in identifying informational principles that account for some of the nonclassical features of quantum physics [35, 36]. However, the quantum treatment of identical particles, one of the most important features of quantum theory insofar as its application is concerned, has hitherto not been analyzed from an informational standpoint. Our purpose here is to provide such an analysis.

Here we show that the symmetrization postulate can be derived in full generality within a fully informational approach: if two particles are indistinguishable on the basis of information obtainable through measurements on them, we refrain from labelling the particles. Our key postulate is the *operational indistinguishability postulate*, which establishes a relation between the theoretical description of two different experiments, positing that the amplitude of a process involving several indistinguishable subsystems (hereafter referred to as *particles*) is determined by the amplitudes of all possible transitions of these particles when treated as distinguishable. Our informational approach allows us to clearly grasp the essential meaning of the symmetrization postulate, and to steer clear of conceptual pitfalls that commonly afflict treatments of this subject [19, 20, 37].

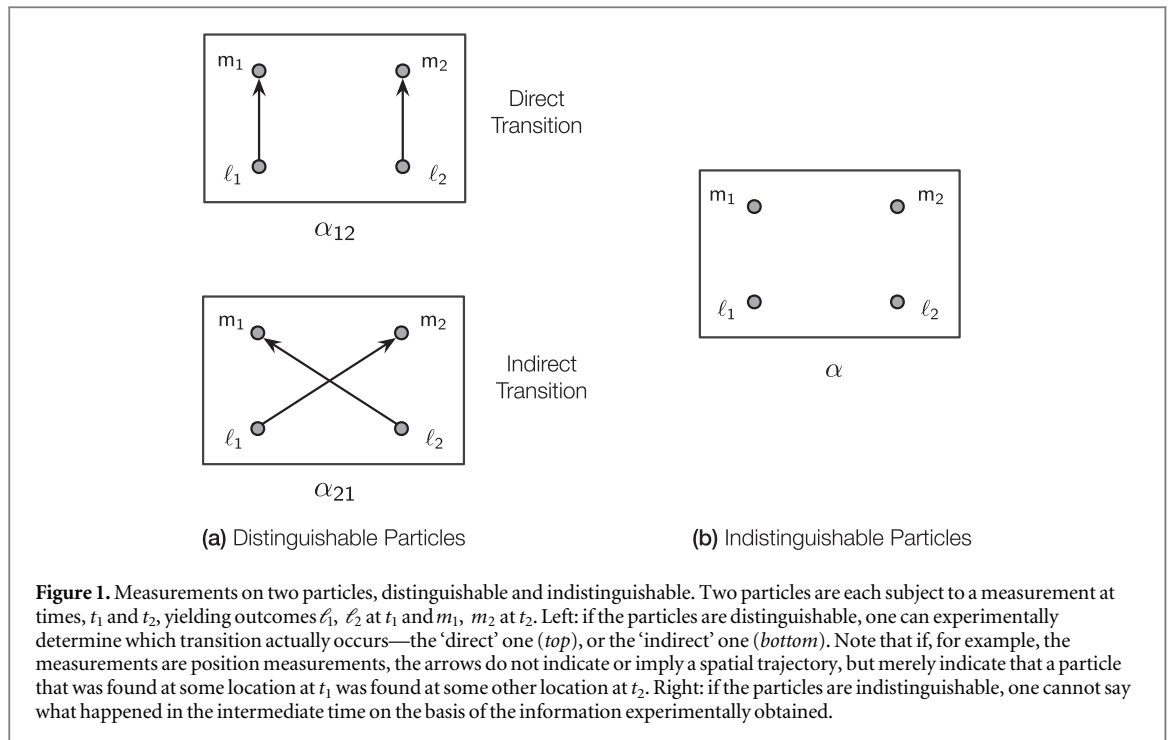
We shall carry out our derivation in the framework of Feynman's formulation of quantum theory, and thereby derive the symmetrization postulate in terms of amplitudes rather than states [38]. We shall then show how the better known state formulation of the symmetrization postulate can be derived from the amplitude form of the symmetrization postulate, and indicate the insights to which this leads.

The derivation harnesses the framework and methodology recently used to derive Feynman's formalism of quantum theory [32], in two key ways. First, we make use of the insight that Feynman's rules are best regarded as establishing a formal relationship between different experiments [32, 39], an insight that lies at the basis of the operational indistinguishability postulate. Second, we employ the simple yet powerful [32, 39] requirement of consistency: if it is possible to compute an amplitude in two different ways, they must agree.

1. Two indistinguishable particles

Consider a quantum system of two particles, each subject to a measurement at successive times t_1 and t_2 . For example, a simultaneous position measurement of both particles can be implemented by a three-dimensional grid of particle detectors which, at each time, yields two position outcomes, one for each particle. We allow for the possibility that these outcomes may take the same (position) value, and we assume (in this idealized setting) that the detector grid is sufficiently extensive and sensitive that it detects precisely two particles at each time, so that neither of the particles goes undetected. Suppose that the measurement performed at t_1 yields outcomes that we shall arbitrarily label ℓ_1 and ℓ_2 , and, similarly, that the measurement at t_2 yields outcomes m_1 and m_2 .

If it is possible to distinguish between the two particles, then there is some measurement (for example, of mass or charge) that we can, in principle, perform on each particle immediately before t_1 and immediately after t_2 which allows us to conclude either that the particle responsible for outcome ℓ_1 is the same as that responsible for m_1 , and similarly that the other particle goes from ℓ_2 to m_2 , which we shall call the 'direct' transition; or that



one particle goes from ℓ_1 to m_2 , and the other from ℓ_2 to m_1 , which we shall call the ‘indirect’ transition¹ (see figure 1).

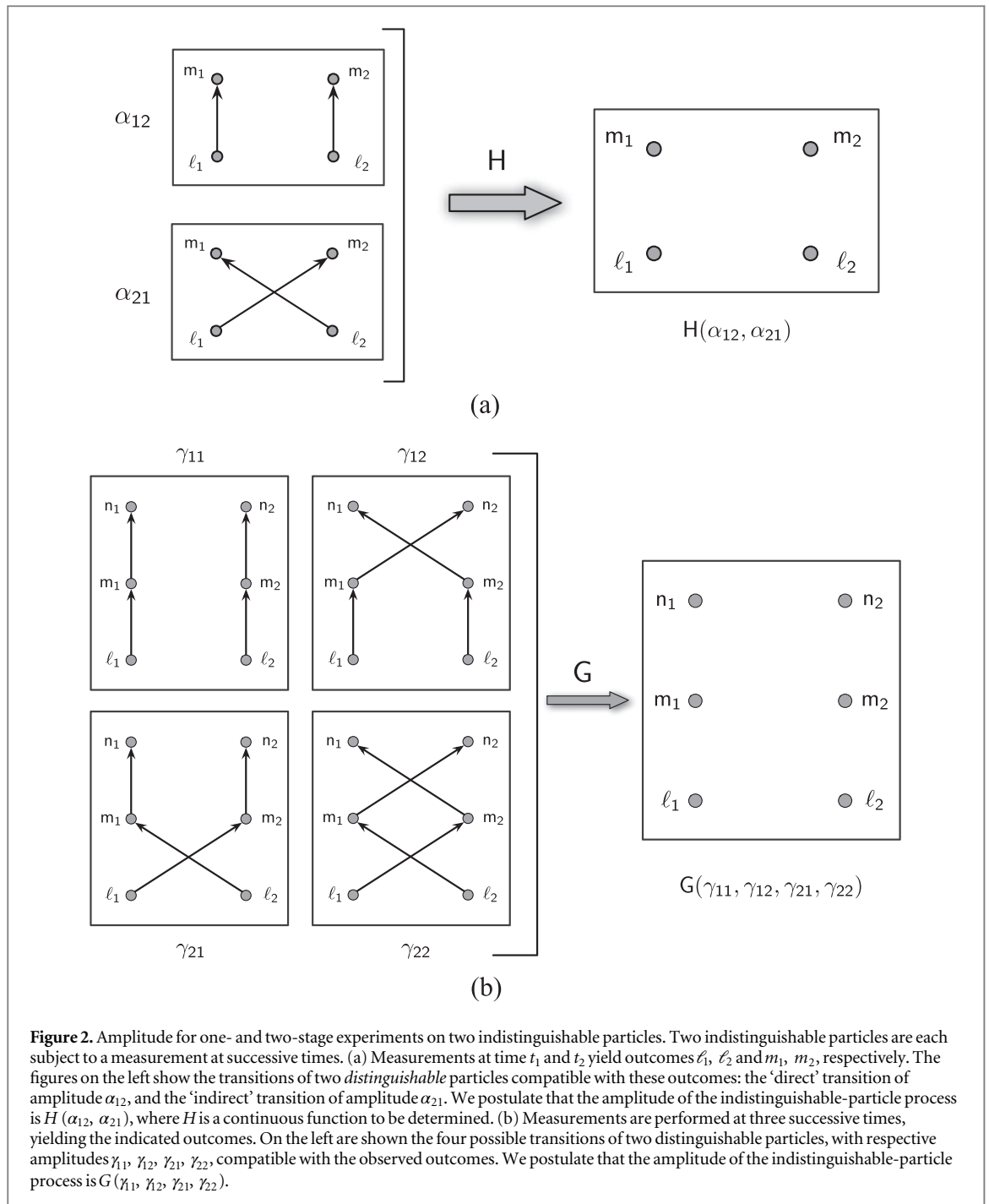
If, however, the particles cannot be distinguished from one another, we cannot draw such a conclusion. Even the conclusion that ‘the particle that generated outcome ℓ_1 was also responsible for outcome m_1 or m_2 , but we do not know which’ is invalid since we cannot verify this statement on account of particle indistinguishability. To nonetheless draw such a conclusion would be tantamount to assuming the separate and continued existence of the particles between observations, which assumption our experience with quantum phenomena warns us against. In particular, in the case of electron diffraction at a double-slit, in the absence of which-way detectors at each slit, we cannot infer from the experimental data that the electron passed through one slit or the other; if we nonetheless insist on drawing this inference, we will discover that our predictions are incorrect—we will be unable to account for the observed diffraction pattern [38, 39].

As we have no way of ‘looking inside’ the indistinguishable-particle process, we must treat it as a black box. As a result, we cannot directly compute the amplitude, α , of the indistinguishable-particle process. Again, electron diffraction is analogous: the transition amplitude of an electron from source to screen in the absence of which-way detectors cannot be computed directly. Instead, it is obtained (via Feynman’s amplitude sum rule) from the sum of the amplitudes of the two possible transitions through the slits with the detectors present, which amplitudes *can* be calculated [39].

In the present case of two particles, then, we seek to relate the amplitude of the indistinguishable-particle transition, α , with the amplitudes, α_{12} and α_{21} , of the ‘direct’ and ‘indirect’ transitions of the two *distinguishable* particles, where each of these particles agree in all their dynamically-relevant properties with the indistinguishable particles. The amplitudes α_{12} and α_{21} can be computed by a standard sum-over-histories from the initial outcomes ℓ_1, ℓ_2 to m_1, m_2 in the Feynman formalism. In the case where the particles are subject to position measurements, α_{12} would consist of a sum over all continuous paths from $(\mathbf{r}'_1, \mathbf{r}'_2)$ to $(\mathbf{r}_1, \mathbf{r}_2)$ in the joint configuration space of the two distinguishable particles, where $\mathbf{r}'_i, \mathbf{r}_i$ are, respectively, the values of outcomes ℓ_i, m_i . Similarly, α_{21} consists of a sum over all paths from $(\mathbf{r}'_1, \mathbf{r}'_2)$ to $(\mathbf{r}_2, \mathbf{r}_1)$. In the Dirac formalism, these amplitudes would be represented as $\langle m_1, m_2 | \ell_1, \ell_2 \rangle$ and $\langle m_2, m_1 | \ell_1, \ell_2 \rangle$.

Now, we cannot relate α to α_{12} and α_{21} via Feynman’s amplitude sum rule since this rule only applies when different experiments are performed on the *same* system, whereas α refers to indistinguishable particles, while α_{12} and α_{21} refer to distinguishable particles. So, in order to derive a connection between α and

¹ The scare quotes (‘direct’, ‘indirect’) reflect the fact that the classification of a distinguishable-particle transition as direct or indirect is dependent upon our choice of labelling of the two outcomes at each time, which choice has so far been left arbitrary. For example, at time t_1 , we have not specified how one chooses which outcome to label ℓ_1 and which to label ℓ_2 . One might wish to posit a labelling convention to remove this arbitrariness, but we do not require such a convention for the derivation of the symmetrization postulate in the Feynman framework. We shall require a labelling convention, however, in order to make the transition to the state formulation of the symmetrization postulate (see section 2).



α_{12} and α_{21} , we must first assume that, as in the case of electron diffraction, such a relation exists. Accordingly, we postulate that

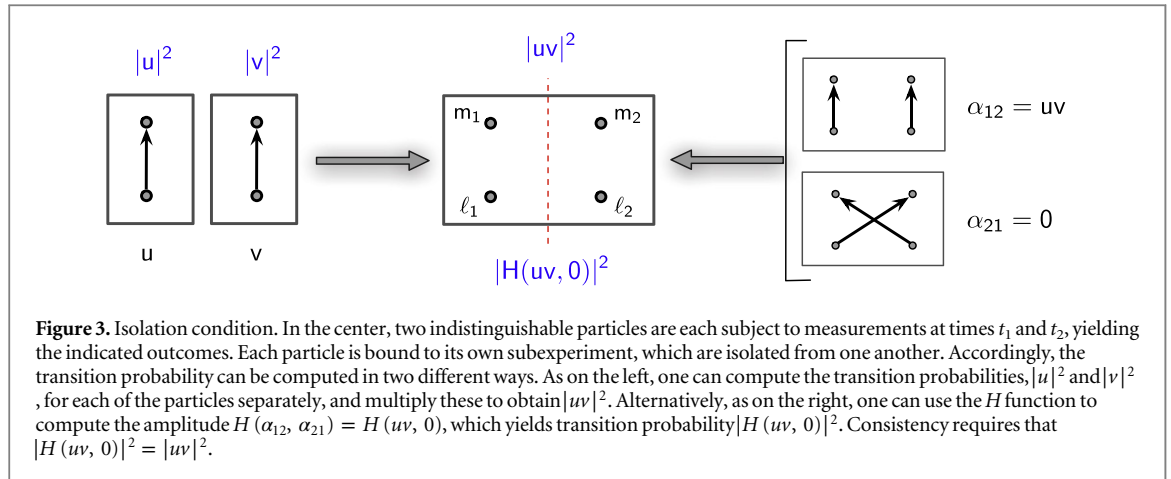
$$\alpha = H(\alpha_{12}, \alpha_{21}), \tag{1}$$

where α lies in the closed unit disc, $\overline{\mathbb{D}}$, in the complex plane, and H is a continuous function, with some domain $S = H^{-1}(\overline{\mathbb{D}}) \subseteq \overline{\mathbb{D}} \times \overline{\mathbb{D}}$, to be determined (see figure 2(a)). Similarly, when two particles are subject to three successive measurements, we postulate that the indistinguishable-particle amplitude, $\gamma \in \overline{\mathbb{D}}$, is given by

$$\gamma = G(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}), \tag{2}$$

where $\gamma_{11}, \gamma_{12}, \gamma_{21}$, and γ_{22} are the amplitudes of the corresponding distinguishable-particle transitions (see figure 2(b)), and G is a function, with some domain $G^{-1}(\overline{\mathbb{D}})$, to be determined. Both of these postulates are special cases of the operational indistinguishability postulate stated above.

Additionally, consider the special case that each particle is known to be bound to a separate experiment, and that the particles do not interact with one another (see figure 3). That is, each particle is confined to its own



subexperiment, and the two subexperiments are physically isolated from one other. In this case, the particle originating at ℓ_1 will later be detected at, say, m_1 , while the particle originating at ℓ_2 will be detected at m_2 . As an example, let one of the particles be an electron that is bound to a proton in a lab on the earth, while the other be, similarly, an electron bound to a proton in a lab on the moon. In such a case, each subexperiment is statistically independent of the other. That is, the outcome probabilities observed in one subexperiment are independent of the outcome probabilities of the other subexperiment, or even of whether or not the other subexperiment is performed.

In this case of isolated subexperiments, we assume, as is conventional, that we can compute the transition probabilities of each subexperiment without regard for the other. We refer to this as the *isolation condition*. In particular, if the amplitude of the transition from ℓ_1 to m_1 is u , then the transition probability is simply $|u|^2$. Likewise, if the amplitude of the transition from ℓ_2 to m_2 is v , then the transition probability is $|v|^2$. Since the subexperiments are statistically independent, the overall transition probability (initial detection at ℓ_1 and ℓ_2 , final detection at m_1 and m_2) is $|u|^2 \cdot |v|^2 = |uv|^2$.

Now, we can also compute this overall transition probability using H . In this special case, $\alpha_{21} = 0$ but $\alpha_{12} = uv$. Hence, the overall transition amplitude is simply $H(uv, 0)$, which yields an overall transition probability of $|H(uv, 0)|^2$. Consistency requires that this agree with the overall transition probability computed above, so that, for all $z \in \overline{\mathbb{D}}$

$$|H(z, 0)| = |z|. \quad (3)$$

Similarly, if the outcomes ℓ_1 and m_2 comprise one subexperiment, and ℓ_2 and m_1 another, and these subexperiments are isolated from one another, then, for all $z \in \overline{\mathbb{D}}$

$$|H(0, z)| = |z|. \quad (4)$$

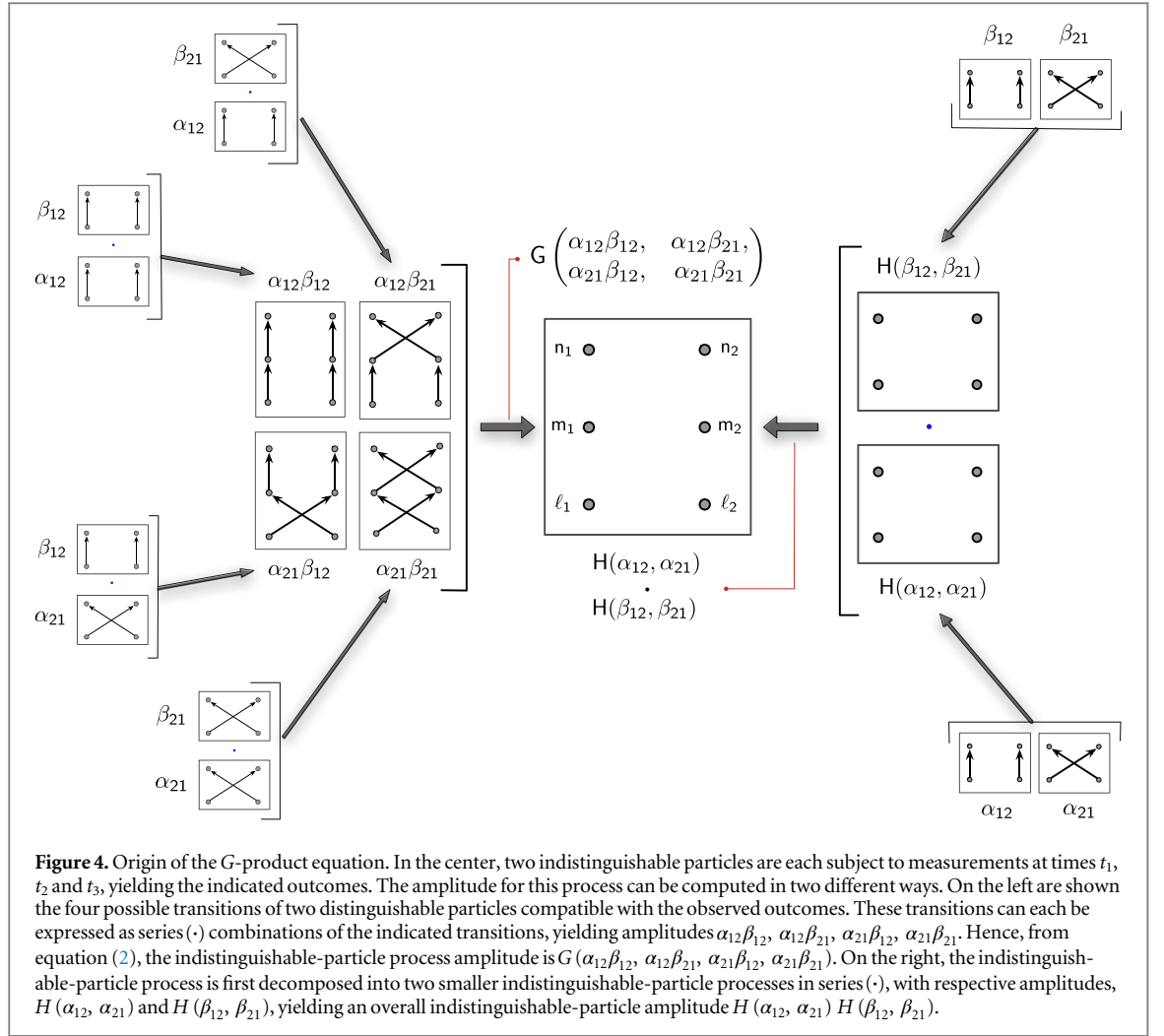
1.1. Formulation of the functional equations

We shall now show that the above assumptions determine the precise form of symmetrization postulate for two particles. The general case of N particles, together with additional detail and background terminology, can be found in appendix A.

To determine G and H , we repeatedly use the idea that, if it is possible to compute an amplitude in two different ways using Feynman's rules for single systems [32, 40] (see also appendix A.2.1) and the assumptions given above, consistency requires that they be equal. Each such call for consistency yields a functional equation. By considering a few special cases, we are able to formulate a set of functional equations that determine G and H .

First, consider a two-stage experiment in which the intermediate outcomes m_1 and m_2 are both atomic [32] (see appendix A for terminology). We can derive the indistinguishable-particle amplitude, γ , in two different ways (see figure 4). Let α_{12} , α_{21} , respectively, be the amplitudes of the direct and indirect transitions in the first stage, and β_{12} , β_{21} the corresponding amplitudes in the second stage. Using the amplitude product rule, we can directly compute the values $\gamma_{11} = \alpha_{12}\beta_{12}$, $\gamma_{12} = \alpha_{12}\beta_{21}$, $\gamma_{21} = \alpha_{21}\beta_{12}$, and $\gamma_{22} = \alpha_{21}\beta_{21}$. Then, using the G -function defined above

$$\gamma = G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21}).$$



Alternatively, we can directly apply the amplitude product rule to the system of identical particles to obtain

$$\gamma = H(\alpha_{12}, \alpha_{21}) H(\beta_{12}, \beta_{21}).$$

Equating these two expressions, we obtain a functional equation, the G -product equation

$$G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21}) = H(\alpha_{12}, \alpha_{21}) H(\beta_{12}, \beta_{21}), \quad (5)$$

valid for all $(\alpha_{12}, \alpha_{21}), (\beta_{12}, \beta_{21}) \in S$.

Next, consider a two-stage experiment in which the outcome of one of the measurements performed at t_2 is a coarse-graining of m_2 and m'_2 , which outcome we denote (m_2, m'_2) (see figure 5). An example of such a coarse-grained outcome would be a Stern–Gerlach measurement where a detector's field of sensitivity encompasses the fields of sensitivity of two detectors, each of which corresponds to a different atomic outcome (see appendix A.1 for further details). Here, α_{12}, α_{21} are respectively the first-stage amplitudes of the direct and indirect transitions compatible with the indistinguishable-particle process through m_2 ; $\alpha'_{12}, \alpha'_{21}$ are similarly the amplitudes corresponding to the process through m'_2 ; and β_{12}, β_{21} are the second-stage amplitudes of the direct and indirect transitions compatible with the indistinguishable-particle process through both m_2 and m'_2 .

The indistinguishable-particle amplitude, $\tilde{\gamma}$, in this case can be computed in two different ways. First, using the amplitude sum and product rules, we can write

$$\tilde{\gamma} = H(\alpha_{12}, \alpha_{21}) H(\beta_{12}, \beta_{21}) + H(\alpha'_{12}, \alpha'_{21}) H(\beta_{12}, \beta_{21}).$$

Alternatively, one can compute $\tilde{\gamma}$ directly from equation (2):

$$\tilde{\gamma} = G((\alpha_{12} + \alpha'_{12})\beta_{12}, (\alpha_{12} + \alpha'_{12})\beta_{21}, (\alpha_{21} + \alpha'_{21})\beta_{12}, (\alpha_{21} + \alpha'_{21})\beta_{21}),$$

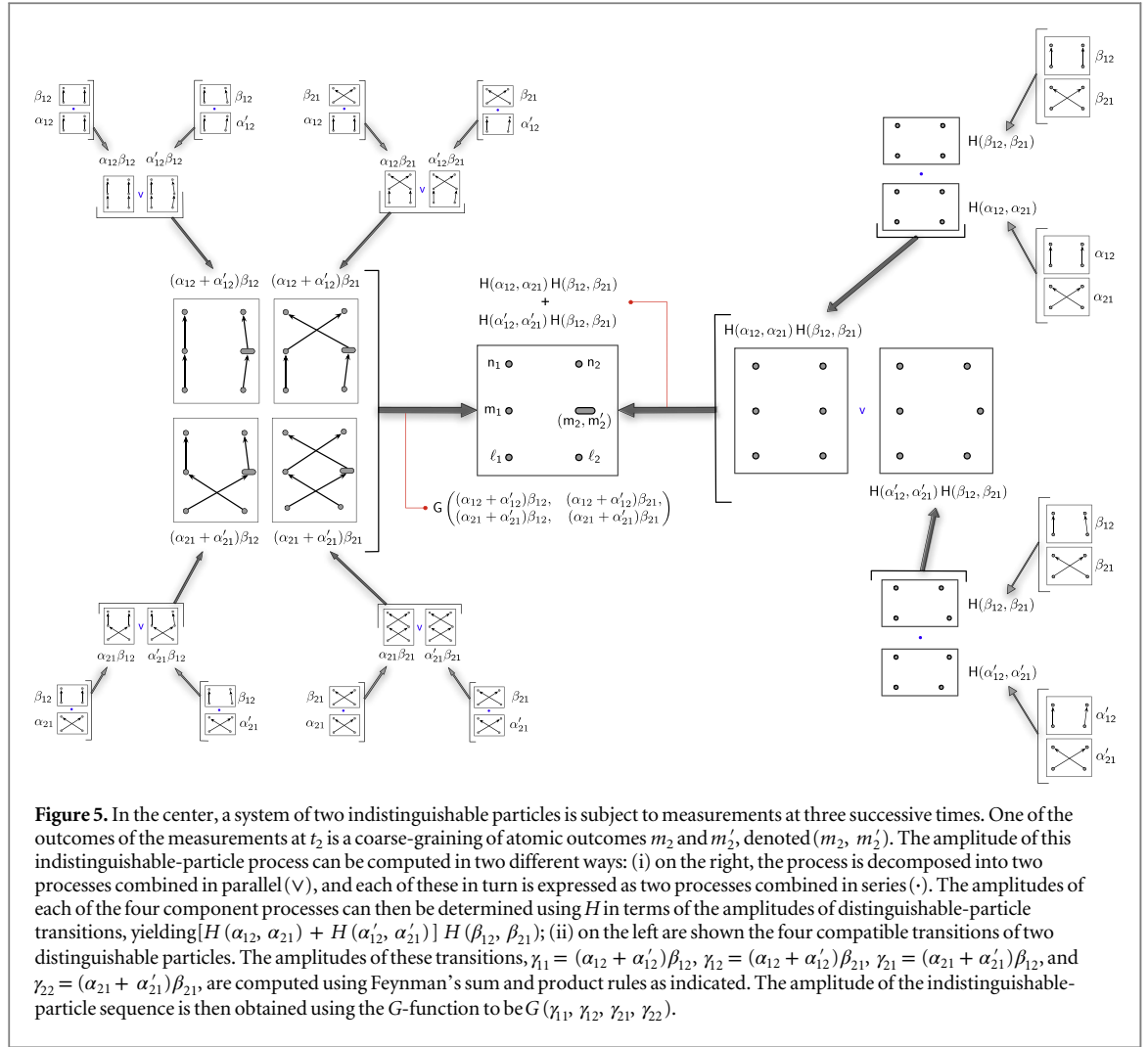


Figure 5. In the center, a system of two indistinguishable particles is subject to measurements at three successive times. One of the outcomes of the measurements at t_2 is a coarse-graining of atomic outcomes m_2 and m'_2 , denoted (m_2, m'_2) . The amplitude of this indistinguishable-particle process can be computed in two different ways: (i) on the right, the process is decomposed into two processes combined in parallel (\vee), and each of these in turn is expressed as two processes combined in series (\cdot). The amplitudes of each of the four component processes can then be determined using H in terms of the amplitudes of distinguishable-particle transitions, yielding $[H(\alpha_{12}, \alpha_{21}) + H(\alpha'_{12}, \alpha'_{21})] H(\beta_{12}, \beta_{21})$; (ii) on the left are shown the four compatible transitions of two distinguishable particles. The amplitudes of these transitions, $\gamma_{11} = (\alpha_{12} + \alpha'_{12})\beta_{12}$, $\gamma_{12} = (\alpha_{12} + \alpha'_{12})\beta_{21}$, $\gamma_{21} = (\alpha_{21} + \alpha'_{21})\beta_{12}$, and $\gamma_{22} = (\alpha_{21} + \alpha'_{21})\beta_{21}$, are computed using Feynman's sum and product rules as indicated. The amplitude of the indistinguishable-particle sequence is then obtained using the G -function to be $G(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22})$.

where we have used the amplitude sum rule to compute the amplitude of the four transitions compatible with the indistinguishable-particle process. Hence, we have the functional equation

$$G\left(\left(\alpha_{12} + \alpha'_{12}\right)\beta_{12}, \left(\alpha_{12} + \alpha'_{12}\right)\beta_{21}, \left(\alpha_{21} + \alpha'_{21}\right)\beta_{12}, \left(\alpha_{21} + \alpha'_{21}\right)\beta_{21}\right) = H\left(\alpha_{12}, \alpha_{21}\right) H\left(\beta_{12}, \beta_{21}\right) + H\left(\alpha'_{12}, \alpha'_{21}\right) H\left(\beta_{12}, \beta_{21}\right), \quad (6)$$

valid for all $(\alpha_{12}, \alpha_{21}), (\alpha'_{12}, \alpha'_{21}), (\beta_{12}, \beta_{21}) \in S$ for which the right-hand side lies in $\overline{\mathbb{D}}$.

Lastly, we consider an experiment on two identical particles in which the sequence of measurements performed on the particles is reversed, and the measurements follow one another immediately in time (see figure 6). As indicated in the figure, using the amplitude conjugation rule, the amplitude can be computed in two different ways, which can be equated to give

$$H^*(\alpha_{12}, \alpha_{21}) = H\left(\alpha_{12}^*, \alpha_{21}^*\right), \quad (7)$$

valid for all $(\alpha_{12}, \alpha_{21}) \in S$.

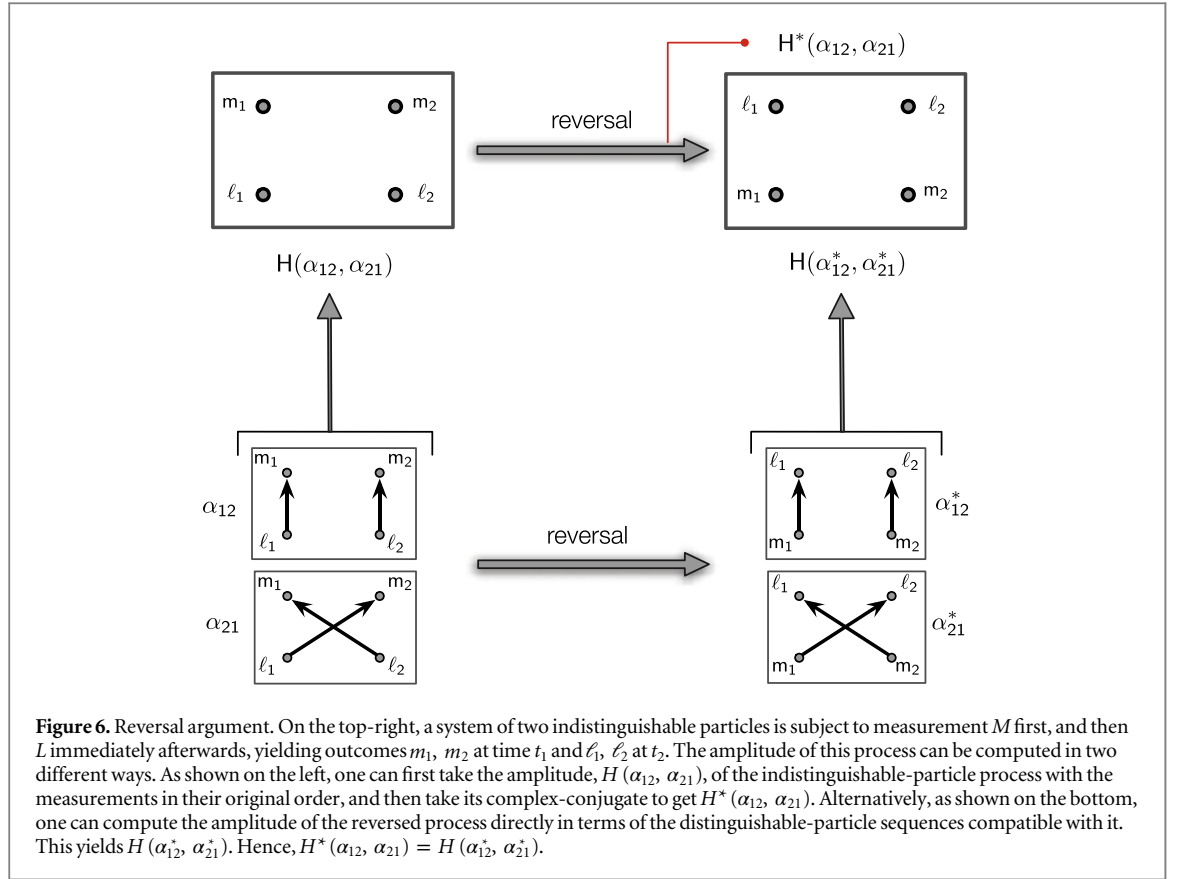
1.2. Solution of the functional equations

We now show that the functional equations (5), (6) and (7), together with equations (3) and (4), lead to the symmetrization postulate for the two particle case.

Using equation (5), the left-hand side of equation (6) can be rewritten to give

$$H\left(\alpha_{12} + \alpha'_{12}, \alpha_{21} + \alpha'_{21}\right) H\left(\beta_{12}, \beta_{21}\right) = H\left(\alpha_{12}, \alpha_{21}\right) H\left(\beta_{12}, \beta_{21}\right) + H\left(\alpha'_{12}, \alpha'_{21}\right) H\left(\beta_{12}, \beta_{21}\right).$$

A nontrivial solution requires that $H(\beta_{12}, \beta_{21})$ cannot be zero for all β_{12}, β_{21} . Hence, we obtain the additivity relation



$$H(\alpha_{12} + \alpha'_{12}, \alpha_{21} + \alpha'_{21}) = H(\alpha_{12}, \alpha_{21}) + H(\alpha'_{12}, \alpha'_{21}), \tag{8}$$

valid for all $(\alpha_{12}, \alpha_{21}), (\alpha'_{12}, \alpha'_{21}), (\alpha_{12} + \alpha'_{12}, \alpha_{21} + \alpha'_{21}) \in S$.

We are now in a position to determine H . First, equation (8) implies that

$$H(\alpha_{12}, \alpha'_{21}) = H(\alpha_{12} + 0, 0 + \alpha'_{21}) = H(\alpha_{12}, 0) + H(0, \alpha'_{21}). \tag{9}$$

Now, equations (3) and (4) establish that $(z, 0), (0, z) \in S$ whenever $z \in \overline{\mathbb{D}}$, and equation (5) then implies that, for all $z \in \overline{\mathbb{D}}$

$$G(0, z, 0, 0) = H(z, 0)H(0, 1) = H(1, 0)H(0, z). \tag{10}$$

Since $H(1, 0) \neq 0$ from equation (3), we can use equation (10) to rewrite equation (9) as

$$H(\alpha_{12}, \alpha_{21}) = H(\alpha_{12}, 0) + \frac{H(0, 1)}{H(1, 0)}H(\alpha_{21}, 0). \tag{11}$$

To determine the form of $H(z, 0)$, we note from equation (8) that, for all $z_1, z_2, z_1 + z_2 \in \overline{\mathbb{D}}$

$$H(z_1 + z_2, 0) = H(z_1, 0) + H(z_2, 0), \tag{12}$$

while equation (5) implies that, for all $z_1, z_2 \in \overline{\mathbb{D}}$

$$H(z_1 z_2, 0)H(1, 0) = H(z_1, 0)H(z_2, 0). \tag{13}$$

Writing $f(z) = H(z, 0)/H(1, 0)$, equations (12) and (13) can be written as a pair of functional equations

$$f(z_1 + z_2) = f(z_1) + f(z_2) \tag{14a}$$

$$f(z_1 z_2) = f(z_1)f(z_2), \tag{14b}$$

where $f(\cdot)$ is a complex-valued function of a complex argument. We are interested in the continuous solutions of these equations for all $z_1, z_2 \in \overline{\mathbb{D}}$ for which $z_1 + z_2 \in \overline{\mathbb{D}}$. As shown in appendix B, these solutions are $f(z) = z, f(z) = z^*$ or $f(z) = 0$. The zero solution is inadmissible since, from its definition, $f(1) = 1$. Therefore

$$H(z, 0) = \begin{cases} H(1, 0) z, \\ H(1, 0) z^*. \end{cases}$$

From equations (3) and (4), we see that $|H(1, 0)| = |H(0, 1)|$; and we note that equation (7) implies that $H(1, 0)$ and $H(0, 1)$ are both real. Therefore

$$H(0, 1) = \pm H(1, 0).$$

Hence, equation (11) becomes

$$H(\alpha_{12}, \alpha_{21}) = \begin{cases} H(1, 0)(\alpha_{12} \pm \alpha_{21}), \\ H(1, 0)(\alpha_{12} \pm \alpha_{21})^*. \end{cases}$$

Since only the transition probability is of physical importance, only the modulus of $H(\alpha_{12}, \alpha_{21})$ is relevant. Hence, without loss of generality, we can take $H(\alpha_{12}, \alpha_{21}) = c(\alpha_{12} \pm \alpha_{21})$, with $c = |H(1, 0)|$. Finally, equation (3) fixes $c = 1$, leaving us with

$$H(\alpha_{12}, \alpha_{21}) = \alpha_{12} \pm \alpha_{21}, \quad (16)$$

where the sign is the only remaining degree of freedom, corresponding to bosons and fermions.

Finally, we note that the above result is based on a particular labelling (ℓ_1, ℓ_2 at t_1 , and so on) of the two outcomes at each time. Now, if the outcome labels at t_1 were swapped, the classification of the transitions into ‘direct’ and ‘indirect’ would also be swapped, inducing the transformation $\alpha_{12} \rightarrow \alpha_{21}$ and $\alpha_{21} \rightarrow \alpha_{12}$, so that one would obtain the amplitude $\alpha_{21} \pm \alpha_{12}$ instead of $\alpha_{12} \pm \alpha_{21}$. But the transformed amplitude $\alpha_{21} \pm \alpha_{12}$ yields the same transition probability as $\alpha_{12} \pm \alpha_{21}$. The same conclusion holds true if the outcome labels at t_2 are swapped. Therefore, the particular choice of labelling at t_1 and t_2 is predictively irrelevant, and so can be chosen arbitrarily without loss of generality. Hence, equation (16) is the final result.

This completes the derivation of the symmetrization postulate for two identical, indistinguishable particles as expressed in Feynman’s formulation of quantum theory.

The derivation for N particles is given in appendix A, and follows a similar line of argument. For N identical particles, there are $N!$ distinguishable-particles transitions that must be considered, each corresponding to some permutation, π , in the symmetric group S_N . Denoting the amplitude of the transition corresponding to π as α_π , we obtain the result

$$H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}) = \sum_{\pi \in S_N} (\text{sgn}(\pi))^\sigma \alpha_\pi, \quad (17)$$

where $\sigma = 0, 1$ is the only remaining degree of freedom, whose value does not depend on the number of particles, N .

2. State formulation of the symmetrization postulate

We shall now briefly discuss how the amplitude formulation of the symmetrization postulate leads to the state formulation. We shall thereby obtain important insight into the interpretation of the latter. For concreteness, we shall restrict attention to two structureless particles moving in one spatial dimension; the generalization to many particles with internal structure moving in d dimensions is straightforward.

Consider two structureless, identical particles that are detected at locations x'_1 and x'_2 at time t_1 , and subsequently at locations x_1 and x_2 at time t_2 . We adopt the convention that, at any given time, the locations are labelled such that the leftmost location is labelled with subscript 1 and the rightmost with subscript 2, so that $x'_1 \leq x'_2$ and $x_1 \leq x_2$. Let $\psi_{\text{ID}}(x_1, x_2)$ be the amplitude for two identical particles to be found at locations x_1, x_2 at time t_2 given that they were found at x'_1, x'_2 at time t_1 . Then, from equation (16)

$$\psi_{\text{ID}}(x_1, x_2) = \psi(x_1, x_2) \pm \psi(x_2, x_1), \quad (18)$$

valid for $x_1 \leq x_2$, where $\psi(x_a, x_b)$ is the amplitude for the transition of two distinguishable particles where one particle (‘particle 1’) goes from x'_1 at t_1 to x_a at t_2 , and the other (‘particle 2’) goes from x'_2 at t_1 to x_b at t_2 .

We note that $\psi_{\text{ID}}(x_1, x_2)$ is normalized over the so-called reduced configuration space of points (x_1, x_2) for which $x_1 \leq x_2$, so that

$$\iint_{x_1 \leq x_2} |\psi_{\text{ID}}(x_1, x_2)|^2 dx_1 dx_2 = 1, \quad (19)$$

whereas $\psi(x_1, x_2)$ is normalized over the full configuration space

$$\iint |\psi(x_1, x_2)|^2 dx_1 dx_2 = 1. \quad (20)$$

We can, however, for convenience, extend ψ_{ID} over the full configuration space by defining the function

$$\tilde{\psi}_{\text{ID}}(x_1, x_2) = \frac{1}{\sqrt{2}} [\psi(x_1, x_2) \pm \psi(x_2, x_1)], \quad (21)$$

valid for all x_1, x_2 , so that $\iint |\tilde{\psi}_{\text{ID}}(x_1, x_2)|^2 dx_1 dx_2 = 1$.

Mathematically, equation (21) is the standard form in which the symmetrization postulate is usually stated. However, as we have derived it from first principles in an informational experimental framework, we are able to make a number of important observations.

First, the normalization equations (19) and (20), together with equation (18), imply that the functions $\psi(x_1, x_2)$ and $\psi(x_2, x_1)$ are orthogonal over the full configuration space. A special case of such functions are product states of the form $\phi(x_1)\varphi(x_2)$ and $\phi(x_2)\varphi(x_1)$, provided that ϕ, φ are mutually orthogonal. This special case is commonly encountered in applications of the symmetrization postulate.

Second, it is commonly asserted that x_i in the expression $\tilde{\psi}_{\text{ID}}(x_1, x_2)$ is the position of particle i , and that therefore $\tilde{\psi}_{\text{ID}}(x_2, x_1)$ represents the state when the identical particles have swapped positions. This assertion forms the basis of many textbook arguments for the symmetrization postulate². However, as one can see from the derivation above, x_1 and x_2 are merely positions of two identical particles, so that the subscripts 1 and 2 label two *locations*, not two *particles*. Moreover, one can see that the use of $\tilde{\psi}_{\text{ID}}$, which is defined over the full configuration space, rather than ψ_{ID} , is merely a convenience—the state of two identical particles is better represented by ψ_{ID} as it cannot be subject to the above-mentioned misinterpretation since it is only defined over the reduced configuration space (as $\psi_{\text{ID}}(x_1, x_2)$ is only defined when $x_1 \leq x_2$, the function ψ_{ID} is undefined on (x_2, x_1) unless $x_1 = x_2$, so that there is no possibility of misinterpreting $\psi_{\text{ID}}(x_2, x_1)$ as meaning that the first particle is at x_2 and the second is at x_1).

Finally, we see that the symmetrization postulate, equation (21), is perhaps better viewed not so much as a constraint (or selection rule) on the space of states that picks out states that can describe indistinguishable particles, but rather as a rule that allows one to construct valid indistinguishable-particle states from certain distinguishable-particle states. In other words, it is a bridge between the two different state spaces, one belonging to distinguishable particles, the other belonging to indistinguishable particles.

3. Discussion

We have shown that there exists a direct path from the assumption that identical particles are indistinguishable to the symmetrization postulate. The derivation rests on the following assumptions:

- (1) the *operational indistinguishability postulate*, which is formalized via equations (1) and (2) for two particles, and via equations (A.5) and (A.6) for N particles; and
- (2) the *isolation condition*, formalized via equations (3) and (4) for two particles, and via equations (A.7) and (A.10) for N particles.

The present derivation differs substantially from the topological approach, both in the nature of its assumptions and in its conclusions. While the topological approach considers only structureless particles, the present derivation makes no assumptions concerning the nature of the particles involved, so that the result is not restricted to structureless particles but applies to any quantum system, such as a system of identical particles with spin, or a system of identical abstract finite dimensional subsystems. In addition, while the topological approach depends critically on the assumption that particles cannot coincide, the present derivation makes no such assumption. Finally, while the topological approach relies on a generalization of Feynman's sum rule for multiply-connected spaces, the present derivation relies only on the standard quantum formalism.

With respect to their conclusions, the topological approach implies that the symmetrization postulate is valid in three spatial dimensions, but fails in two dimensions, allowing for a one-parameter family of so-called anyonic behaviors. In contrast, the present derivation implies that the validity of the symmetrization postulate is unrelated to the dimension of space, and, in particular, that the symmetrization postulate is valid in two dimensions.

² See, for example, [41], chapter 9.

We emphasize that our conclusion (that, *contra* the topological approach, the symmetrization postulate is valid in two dimensions) is not at odds with the experiment observation of anyonic behaviour in such phenomena as the fractional Hall effect. Although such phenomena are often believed to confirm the idea that the symmetrization postulate breaks down in two dimensions, anyonic behaviour arises generically when particles are confined to two dimensions if the particles are unable to coincide, irrespective of whether the particles are identical or nonidentical [42, 43]. Hence, if one is dealing with a system of identical particles that cannot coincide, and these particles are confined to two dimensions, then the anyonic behaviour will make itself felt through the α_π . That is, the presence of anyonic behaviour does not preclude the validity of the symmetrization postulate in two dimensions.

To understand this point in more detail, let us briefly consider two structureless, distinguishable particles, with positions \mathbf{r}_1 and \mathbf{r}_2 , which cannot coincide and are confined to two dimensions. In this case, the relevant configuration space is the so-called relative configuration space of $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$ where the origin, $\mathbf{R} = \mathbf{0}$, is excluded as the particles cannot coincide. Owing to the exclusion of the origin, two paths from point \mathbf{R}_A to \mathbf{R}_B in this space cannot always be deformed into one another, and the set of all paths from \mathbf{R}_A to \mathbf{R}_B breaks into a countable infinity of so-called homotopy classes, each class indexed by the winding number of the paths it contains. By Schulman's generalized sum rule for multiply-connected spaces [16], using the results of [3], one finds that the amplitude $\tilde{\alpha}$ for the transition from \mathbf{R}_A to \mathbf{R}_B consists of an infinite sum of so-called partial amplitudes

$$\tilde{\alpha} = \sum_{n=-\infty}^{\infty} e^{in\phi} K^{(n)}, \quad (22)$$

where each such partial amplitude, $K^{(n)}$, is the sum of the amplitudes of all paths in homotopy class n , where n is the winding number of all paths in the class. The single degree of freedom, ϕ , in the above expression, which arises because of the exclusion of particle coincidence, is responsible for the so-called anyonic behavior.

This degree of freedom naturally carries forward to the case where the particles are identical. The transition for two identical particles from \mathbf{R}_A to \mathbf{R}_B in the reduced configuration space is given by $\alpha = \alpha_{12} \pm \alpha_{21}$. In this expression, α_{12} is the amplitude of the transition of two distinguishable particles from \mathbf{R}_A to \mathbf{R}_B , and hence has the form of equation (22). The amplitude α_{21} for the transition from \mathbf{R}_A to $-\mathbf{R}_B$ similarly involves ϕ [44]. Hence, the identical-particle amplitude $\alpha = \alpha_{12} \pm \alpha_{21}$ also involves a single additional degree of freedom, ϕ , which arises from the exclusion of particle coincidence, and it is this degree of freedom which is responsible for anyonic behavior. Hence, experimental evidence for anyonic behaviour does not require renunciation of the symmetrization postulate, but only exclusion of particle coincidence.

More broadly, the present derivation is a natural extension and application of the ideas previously used to derive Feynman's rules of quantum theory. This demonstrates that the symmetrization postulate is comparable in nature and reliability to the core quantum formalism, contrary to what has previously appeared to be the case.

As we have described previously [39], the methodology used to derive Feynman's rules has been previously used to derive [45] the formalism of probability theory starting from Boolean logic, and has recently been applied to yield important insights in other domains [46]. The successful application of this methodology to the symmetrization postulate leads us to anticipate that this methodology may provide a valuable tool in understanding and constructing physical theory.

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Appendix A. System of many identical subsystems

In this section, we present the derivation of the symmetrization postulate for N indistinguishable subsystems. This derivation closely follows the derivation for two indistinguishable subsystems given in the main text. For reasons of precision and clarity, we explicitly employ the the operational framework and notation used in [32]. For completeness, we first summarize the key features of this framework, and then describe the extensions we need in order to describe composite systems. We then summarize how one can describe Feynman paths and Feynman's amplitude rules in this operational framework. Having laid this foundation, we present the derivation itself.

A.1. Operational framework

As explained in [32, 47], the measurements and interactions which can be employed in a given experiment must be carefully circumscribed if they are to lead to a well-defined theoretical model. The overarching requirement is that, in the given experiment, the outcome probabilities of any measurement but the first are independent of any interactions with the system prior to the first measurement, a requirement we refer to as *experimental closure*.

Intuitively, the first measurement ‘screens off’ the prior history of the system, rendering this prior history irrelevant insofar as making predictions about the outcomes of subsequent measurements in the experiment is concerned. The requirement of experimental closure naturally leads to the following definitions and constraints.

An experimental set-up is defined by specifying a source of physical systems, a sequence of measurements to be performed on a system on a run of the experiment, and the interactions with the system which occur between the measurements. In a run of an experiment, a physical system from the source passes through a sequence of measurements L, M, N, \dots , which respectively yield outcomes ℓ, m, n, \dots at times t_1, t_2, t_3, \dots . These outcomes are summarized in the *measurement sequence* $[\ell, m, n, \dots]$. To avoid notational clutter, the measurements that yield these outcomes, the times at which these outcomes occur, and the nature of the system that is under examination, are all left implicit in the notation, and so must be inferred from the context. In between these measurements, the system may undergo interactions with the environment.

Over many runs of the experiment, the experimenter will observe the frequencies of the various possible measurement sequences, from which one can (using Bayes’ rule) estimate the probability associated with each sequence. We define the probability $P(A)$ associated with sequence $A = [\ell, m, n, \dots]$ as the probability of obtaining outcomes m, n, \dots conditional upon obtaining ℓ

$$P(A) = \Pr(m, n, \dots \mid \ell). \quad (\text{A.1})$$

A particular outcome of a measurement is either *atomic* or *coarse-grained*. An atomic outcome is one that cannot be more finely divided in the sense that the detector whose output corresponds to the outcome cannot be sub-divided into smaller detectors whose outputs correspond to two or more outcomes. We refer to a detector whose firing generates an atomic outcome as an atomic detector. An example of atomic outcomes are the two possible outcomes of a Stern–Gerlach measurement performed on a silver atom. A coarse-grained outcome is one that results from the firing of a coarse-grained detector, namely a detector whose field of sensitivity encompasses those of two or more atomic detectors. To be clear, a coarse-grained outcome does not mean that one has made a fine-grained measurement, thereby obtaining one of N possible outcomes, and one has then applied a many-to-one map yielding one of less than N possible outcomes, a process that discards information and could be regarded as an *informational* coarse-graining. Rather, a coarse-grained outcome is that due to the firing of a detector that is a *physical* coarse-graining of two or more detectors.

Abstractly, if measurement \bar{M} has an outcome which results from a detector which is a coarse-graining of atomic detectors whose outcomes are labelled 1 and 2 of measurement M , this outcome of \bar{M} is labelled (1, 2). This notational convention naturally extends to coarse-graining of more than two atomic detectors. In general, if all of the possible outcomes of a measurement are atomic, we shall call the measurement itself atomic. Otherwise, we say it is a coarse-grained measurement.

It is important that all of the measurements that are employed in an experimental set-up come from the same *measurement set*, \mathcal{M} , or are coarsened versions of measurements in this set. This ensures that all the measurements are probing the same aspect of the system. The set is operationally defined as follows. Measurement M forms a *measurement pair* with L if in both (a) experiment 1, where measurement M is performed on a given system immediately after it has been prepared using L , and (b) experiment 2, where measurement L is used to prepare the given system, and measurement M is performed immediately afterwards, we have *experimental closure*, namely that the outcome probabilities of the final measurement in both instances are independent of interactions with the system prior to the preparation. The measurement set, \mathcal{M} , containing M is the set of all measurements that form a measurement pair with M . Interactions that occur in the period of time *between* measurements are likewise selected from a set, \mathcal{I} , of possible interactions, which are such that they preserve closure when performed between any pair of measurements from \mathcal{M} . Experimental closure requires that the first measurement, M , in an experiment is atomic; the remaining measurements must either lie in the same measurement set as M or be coarsened versions of measurements in this set.

A.1.1. Composite systems. The operational framework sketched above is concerned with measurements that probe the entire system. But, in the case of a composite system, it is possible to perform measurements that only probe particular subsystems of the system. For example, consider a system subject to position measurements at successive times. If, at each of these times, two clicks are registered, we say that the system is a composite system of two particles.

More formally, consider an experimental arrangement in which, at each time, one measurement is performed on each of the two subsystems of a composite system. If, at time t_1 , measurements L_1 are performed

on one subsystem and L_2 on the other, and simultaneously yield outcomes ℓ_1 and ℓ_2 , respectively, we shall notate the outcome of the two measurements as $(\ell_1: \ell_2)$. If the measurement L_1 and L_2 are performed at time t_1 , and measurements M_1 and M_2 are subsequently performed at t_2 , the resulting measurement sequence can accordingly be written as $C = [(\ell_1: \ell_2), (m_1: m_2)]$. To ensure closure, the measurements L_1 , L_2 , M_1 , and M_2 must all lie in the same measurement set.

If it is possible to carry out measurements immediately before t_1 and after t_2 whose outcomes together determine whether the subsystem detected at M_1 originated at L_1 or at L_2 , we say that these subsystems are distinguishable. Otherwise we say they are indistinguishable, in which case the statement that subsystem 1 originated at L_1 and transitioned to, say, M_1 , is not, in general, operationally meaningful.

A.2. Operationalization of Feynman's paths and Feynman's rules

Consider an experimental set-up in which a physical system is subject to successive measurements L , M , N at successive times t_1 , t_2 , t_3 , with there possibly being interactions with the system in the intervals between those measurements. Here and subsequently, we assume that the measurements and interactions in any such set-up are selected according to the constraints described above. We summarize the outcomes obtained in a given run of the experiment as the *sequence* $C = [\ell, m, n]$. This is the operational counterpart to a Feynman 'path'.

We now wish to formalize the idea that set-ups are interrelated in particular ways. In [32], we considered three such relationships. First, the above set-up could be viewed as a *series concatenation* of two experiments, the first in which measurements L and M occur at times t_1 and t_2 , yielding the sequence $A = [\ell, m]$, and the second in which measurements M and N occur at times t_2 and t_3 , yielding $B = [m, n]$. In order to ensure that experimental closure is satisfied in the second experiment, measurement M must be atomic. Formally, we express this concatenation as

$$C = A \cdot B, \quad (\text{A.2})$$

where \cdot is the *series* combination operator. More generally, the series operator can be used to concatenate two sequences provided their initial and final measurements are atomic, and the final measurement and outcome of the first sequence is the same as the initial measurement and outcome of the second sequence.

Second, one can consider a set-up which is identical to the one above, except that detectors corresponding to outcomes m and m' of M have been coarse-grained, so that one obtains the sequence $E = [\ell, (m, m'), n]$. Formally, we express the relationship of this sequence to the sequences $C = [\ell, m, n]$ and $D = [\ell, m', n]$ as

$$E = C \vee D, \quad (\text{A.3})$$

where \vee is the *parallel* combination operation. More generally, the parallel operator can combine any two sequences which are identical except for differing in the outcome of a single measurement in the set-up, provided that this measurement is not the initial or final measurement.

Third and finally, we can consider a set-up which is identical to the one above, except that the sequence of measurements has been reversed, so that measurements N , M and L occur at times t_1 , t_2 and t_3 , respectively, and yield the sequence $F = [n, m, \ell]$. We express the relationship of this sequence to C as

$$F = \overleftarrow{C}, \quad (\text{A.4})$$

where $\overleftarrow{}$ is the *reversal* operator.

A.2.1. Feynman's rules in operational form. From the above definitions, it follows that the operators \cdot and \vee satisfy several symmetry relations, to which we collectively refer to as an *operational logic*. In [32], it is shown that Feynman's rules are the unique pair-valued representation of this logic consistent with a few additional assumptions. Writing $z(X)$ for the complex-valued *amplitude* that represents sequence X , one finds

$$\begin{aligned} z(A \vee B) &= z(A) + z(B) && \text{(amplitude sum rule)} \\ z(A \cdot B) &= z(A) \cdot z(B) && \text{(amplitude product rule)} \\ z(\overleftarrow{A}) &= z^*(A) && \text{(amplitude conjugation rule)} \\ P(A) &= |z(A)|^2. && \text{(probability rule)} \end{aligned}$$

These are Feynman's rules for measurements on single quantum systems. We remark that the amplitude conjugation rule only holds in the special case where the measurements in sequence A immediately follow one another in time.

A.3. Derivation

The derivation for N indistinguishable subsystems closely follows the derivation for two indistinguishable subsystems given in the main text. The key postulates are as follows. Consider an experiment involving $N > 1$

indistinguishable subsystems, with measurements L_1, L_2, \dots, L_N performed at time t_1 , and M_1, M_2, \dots, M_N performed at t_2 . We postulate that the amplitude of the sequence $A = [\ell, m]$, where $\ell \equiv (\ell_1: \ell_2: \dots: \ell_N)$ and $m \equiv (m_1: m_2: \dots: m_N)$, is a function of the amplitudes of the transitions of N distinguishable particles compatible with A . There are $N!$ such transitions, each corresponding to some permutation, π , in the symmetric group S_N : in a transition corresponding to π , the subsystem that was measured at L_j is later found at $M_{\pi(j)}$, for $j = 1, 2, \dots, N$. We notate this transition as $\pi_1 \rightarrow \pi$, where π_1 is the identity permutation here and subsequently. Accordingly, we write

$$z(A) = H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}), \quad (\text{A.5})$$

where the π_i are the $N!$ distinct permutations in S_N , and α_π is the amplitude of the distinguishable-particle transition $\pi_1 \rightarrow \pi$.

Similarly, for three stages of measurement, with each stage consisting of N measurements, we postulate that the sequence $C = [\ell, m, n]$, where $n \equiv (n_1: n_2: \dots: n_N)$, has amplitude

$$z(C) = G(\Gamma), \quad (\text{A.6})$$

where $\Gamma_{ij} = \gamma_{ij}$ is the amplitude of the transition $\pi_1 \rightarrow \pi_i \rightarrow \pi_j$.

A.3.1. Isolation condition. Consider an experiment involving N indistinguishable subsystems, with measurements L_1, L_2, \dots, L_N performed at time t_1 , and M_1, M_2, \dots, M_N performed at t_2 . If a subset of these subsystems is bound to an isolated subexperiment, we shall assume, as is conventional, that we can compute the transition probability for the subexperiment without regard for the subsystems that are not part of the subexperiment. This is the isolation condition.

We shall consider two special cases. In the first, each subsystem is localized to its own subexperiment (see figure 3). As an example, we may have N electrons, each localized to a widely separated hydrogen atom. In particular, suppose that, for $j = 1, \dots, N$, and some permutation π_j , the subexperiment involving measurements L_j and $M_{\pi_j(j)}$ is isolated. That is, with certainty, the subsystem originating at outcome ℓ_j of measurement L_j will later be detected at some outcome, $m_{\pi_j(j)}$, of measurement $M_{\pi_j(j)}$, and the probability of the transition from ℓ_j to $m_{\pi_j(j)}$ is independent of the transition probabilities of the $N - 1$ other subexperiments.

By the isolation condition, we can compute the transition probability of the transition from ℓ_j to $m_{\pi_j(j)}$ without regard to the other subexperiments. If the transition ℓ_j to $m_{\pi_j(j)}$ has amplitude u_j for $j = 1, \dots, N$, then the corresponding transition probability is $|u_j|^2$. Hence, the probability of the transition from ℓ to m is

$$\Pr(m|\ell) = \prod_{j=1}^N |u_j|^2 = |z|^2, \quad (\text{A.7})$$

where $z = u_1 u_2 \dots u_N$. We can also compute this transition probability using equation (A.5) with amplitude $\alpha_{\pi_k} = z \delta_{\pi_i, \pi_k}$. Now, let us define $J_i(z) \equiv H(0, \dots, 0, z, 0, \dots, 0)$ where z appears as the i th argument of H . Then, the amplitude of the transition is $J_i(z)$, so that

$$\Pr(m|\ell) = |J_i(z)|^2. \quad (\text{A.8})$$

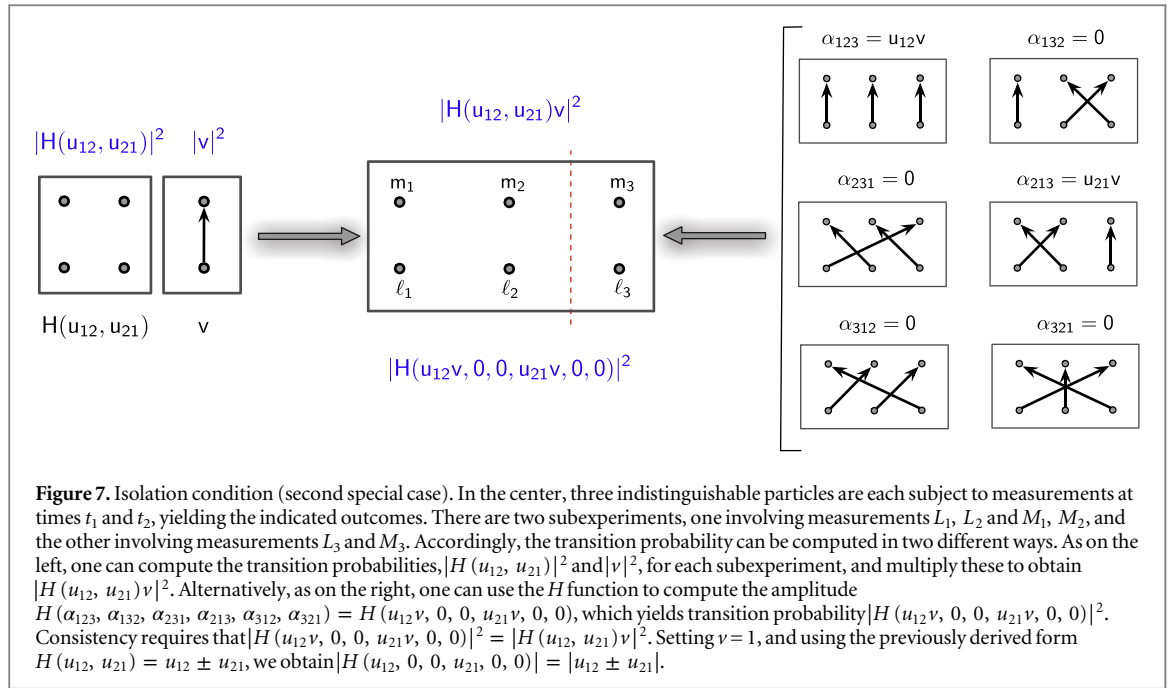
Consistency of equations (A.7) and (A.8) requires that, for all i ,

$$|J_i(z)| = |z|. \quad (\text{A.9})$$

In the second special case, we have two subsystems localized to their own subexperiment, while each of the other subsystems are localized to their own subexperiment (see figure 7). As an example, we may have two electrons bound to a helium atom, while the other $N - 2$ electrons are each bound to their own hydrogen atom. Let us suppose that the measurements L_a, L_b and $M_{\pi(a)}, M_{\pi(b)}$ form a subexperiment, while L_j and $M_{\pi(j)}$ for $j \neq a, b$, form a subexperiment, where π is some permutation.

By the isolation condition, we can compute the transition probability from ℓ to m in two different ways. First, the transition amplitude for the subexperiment involving L_a, L_b is $H(u_{12}, u_{21})$, where u_{12}, u_{21} are the amplitudes of the direct and indirect transitions of two distinguishable systems from L_a, L_b to $M_{\pi(a)}, M_{\pi(b)}$. Therefore, the transition probability for this subexperiment is $|H(u_{12}, u_{21})|^2$. And let us suppose that the amplitudes of the transitions of remaining $N - 2$ subexperiments are all unity, so that their transition probabilities are all unity. Then, the overall transition probability from ℓ to m is $|H(u_{12}, u_{21})|^2$.

Second, we can use H for N particles (equation (A.5)) with $\alpha_{\pi_i} = u_{12} \delta_{\pi, \pi_i} + u_{21} \delta_{\pi', \pi_i}$, where $\pi' = \tau \pi$ and τ is the transposition $(\pi(a), \pi(b))$, to obtain the amplitude $H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}})$, and hence the transition probability $|H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}})|^2$ for the overall transition. Consistency requires that



Using equation (16), this becomes

$$|H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}})| = |u_{12} + (-1)^\sigma u_{21}|, \tag{A.11}$$

where $\sigma = 0$ or 1 , with $\alpha_{\pi_i} = u_{12}\delta_{\pi, \pi_i} + u_{21}\delta_{\pi', \pi_i}$.

A.3.2. The G-product equation. In the case where the outcomes at t_2 are atomic, we can compute the amplitude of the sequence $C = [\ell, m, n]$ in two different ways. First, writing α_π as the amplitude of the distinguishable-particle transition $\pi_1 \rightarrow \pi$ compatible with $[\ell, m]$ and similarly β_π as the amplitude of the transition $\pi_1 \rightarrow \pi$ compatible with $[m, n]$, we can directly use equation (A.6) to get

$$z(C) = G(\Gamma),$$

where $\Gamma_{ij} = \alpha_{\pi_i}\beta_{\pi_j}$ is the amplitude of the distinguishable-particle transition $\pi_1 \rightarrow \pi_i \rightarrow \pi_j$, computed using the amplitude product rule. Alternatively, we can write

$$C = [\ell, m] \cdot [m, n]$$

so that

$$z(C) = H(\alpha_{\pi_1}, \dots, \alpha_{\pi_{N!}}) H(\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}).$$

Equating these two results, we obtain a functional equation, the G-product equation

$$G(\Gamma) = H(\alpha_{\pi_1}, \dots, \alpha_{\pi_{N!}}) H(\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}). \tag{A.12}$$

A.3.3. Coarse graining, and the additivity relation for H. Next, consider an experiment with measurements at times t_1, t_2 and t_3 where measurement M_q , for some $q \in \{1, \dots, N\}$, is coarse-grained, and has an outcome $\tilde{m}_q = (m_q^{(1)}, m_q^{(2)})$. Suppose that the sequence $C = [\ell, m, n]$, where $m \equiv (m_1: m_2: \dots: \tilde{m}_q: \dots: m_N)$, is obtained. We can compute its amplitude in two ways.

Let $\alpha_{\pi_i}^{(r)}$ be the amplitude of the transition $\pi_1 \rightarrow \pi_i$ compatible with $[\ell, (m_1: m_2: \dots: m_q^{(r)}: \dots: m_N)]$, and likewise $\beta_{\pi_j}^{(r)}$ the amplitude of the transition $\pi_1 \rightarrow \pi_j$ compatible with $[(m_1: m_2: \dots: m_q^{(r)}: \dots: m_N), n]$. We restrict attention to the special case where $\beta_{\pi_j}^{(r)} = \beta_{\pi_j}$. Then, we can directly apply equation (A.6) to get

$$z(C) = G(\Gamma),$$

where $\Gamma_{ij} = (\alpha_{\pi_i}^{(1)} + \alpha_{\pi_i}^{(2)})\beta_{\pi_j}$ is computed using the amplitude sum and product rules. Using the G-product equation (equation (A.12)), this can be written

$$z(C) = H(\alpha_{\pi_1}^{(1)} + \alpha_{\pi_1}^{(2)}, \dots, \alpha_{\pi_{N!}}^{(1)} + \alpha_{\pi_{N!}}^{(2)}) H(\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}).$$

Alternatively, we can write the sequence C as

$$C = \bigvee_{r=1}^2 [\ell, (m_1: m_2: \dots: m_q^{(r)}: \dots: m_N), n]$$

whose amplitude is given using the amplitude sum rule by

$$z(C) = \sum_{r=1}^2 H(\alpha_{\pi_1}^{(r)}, \alpha_{\pi_2}^{(r)}, \dots, \alpha_{\pi_{N!}}^{(r)}) H(\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}).$$

If $H(\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}) \neq 0$ for some values of $\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}$, we can equate these two expressions for $z(C)$ to obtain the additivity relation

$$H(\alpha_{\pi_1}^{(1)} + \alpha_{\pi_1}^{(2)}, \alpha_{\pi_2}^{(1)} + \alpha_{\pi_2}^{(2)}, \dots, \alpha_{\pi_{N!}}^{(1)} + \alpha_{\pi_{N!}}^{(2)}) = \sum_{r=1}^2 H(\alpha_{\pi_1}^{(r)}, \alpha_{\pi_2}^{(r)}, \dots, \alpha_{\pi_{N!}}^{(r)}). \quad (\text{A.13})$$

The case $H(\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}) = 0$ for all $\beta_{\pi_1}, \dots, \beta_{\pi_{N!}}$ is the trivial solution that implies that every transition of N indistinguishable particles has zero probability. This solution can therefore be discarded.

A.3.4. Simplification of expression for H . From the additivity relation equation (A.13), it follows that

$$H(\alpha_{\pi_1} + 0, \alpha_{\pi_2} + 0, \dots, \alpha_{\pi_{N!}} + 0, 0 + \alpha_{\pi_{N!}}) = H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}, 0) + H(0, 0, \dots, 0, \alpha_{\pi_{N!}}).$$

Iteratively applying the additivity relation, we obtain

$$\begin{aligned} H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}) &= H(\alpha_{\pi_1}, 0, \dots, 0) + H(0, \alpha_{\pi_2}, 0, \dots, 0) + \dots + H(0, 0, \dots, 0, \alpha_{\pi_{N!}}) \\ &= J_1(\alpha_{\pi_1}) + J_2(\alpha_{\pi_2}) + \dots + J_{N!}(\alpha_{\pi_{N!}}). \end{aligned} \quad (\text{A.14})$$

We can interrelate the $J_i(\cdot)$ using the G -product equation (equation (A.12)). Taking $I_{ij} = \delta_{ik} \delta_{j\ell} z$, and noting that equation (A.9) ensures that all $z \in \overline{\mathbb{D}}$ lie in the domain of J_i for all i , the G -product equation implies

$$G(\mathbf{I}) = J_k(z) J_\ell(1) = J_k(1) J_\ell(z).$$

Let us write $J(z) \equiv J_1(z)$. Since $J(1) \neq 0$ from equation (A.9), we can use the above expression for $G(\mathbf{I})$ to rewrite equation (A.14) solely in terms of $J(z)$ and the constants $J_i(1)$:

$$H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}) = \frac{1}{J(1)} \sum_{i=1}^{N!} J_i(1) J(\alpha_{\pi_i}).$$

To fix the form of $J(z)$, we note that, from equation (A.13)

$$J(z_1 + z_2) = J(z_1) + J(z_2), \quad (\text{A.15})$$

while equation (A.12) implies

$$J(z_1 z_2) J(1) = J(z_1) J(z_2). \quad (\text{A.16})$$

Writing $J(z) = J(1) f(z)$ transforms equations (A.16) and (A.15) into equations (14a) and (14b). As in the two-particle case in the main text, we are interested in the continuous solutions of these equations for all those z_1, z_2 which correspond to probabilities in $[0, 1]$ for which $z_1 + z_2, f(z_1), f(z_2)$ and $f(z_1 + z_2)$ also correspond to probabilities in $[0, 1]$. Now, due to equation (A.9), $J(z)$, and so also $f(z)$, lies in the closed unit disc, $\overline{\mathbb{D}}$, in the complex plane whenever $z \in \overline{\mathbb{D}}$. Therefore, we seek the continuous solutions of equations (14a) and (14b) for all $z_1, z_2 \in \overline{\mathbb{D}}$ for which $z_1 + z_2 \in \overline{\mathbb{D}}$. As shown in appendix B, these solutions are $f(z) = z, f(z) = z^*$ or $f(z) = 0$. The zero solution is inadmissible since, from its definition, $f(1) = 1$. Hence

$$J(z) = \begin{cases} J(1) z, \\ J(1) z^*. \end{cases}$$

Therefore, writing $J_i(1)$ as $Q(\pi_i)$

$$H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}) = \begin{cases} \sum_{\pi \in S_N} Q(\pi) \alpha_\pi, \\ \sum_{\pi \in S_N} Q(\pi) \alpha_\pi^*. \end{cases}$$

(a) *Determination of values of the $Q(\pi)$.* To determine the values of $Q(\pi)$ for all possible permutations $\pi \in S_n$, we first show that $Q(\pi) = \pm 1$, and then show that $Q(\pi) = Q(\pi')$ whenever π and π' are both odd or both even.

(i) Establishing that $Q(\boldsymbol{\pi}) = \pm 1$. The reverse, \overleftarrow{A} , of sequence $A = [\ell, m]$, where $\ell \equiv (\ell_1: \ell_2: \dots: \ell_N)$ and $m \equiv (m_1: m_2: \dots: m_N)$, can be computed in two distinct ways (see figure 6) if the measurements immediately follow one another in time. Equating the resulting expressions, we obtain

$$H(\alpha_{\pi_1}^*, \alpha_{\pi_2}^*, \dots, \alpha_{\pi_{N!}}^*) = H^*(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}).$$

In particular, for all $\boldsymbol{\pi}$

$$Q(\boldsymbol{\pi}) = Q^*(\boldsymbol{\pi}),$$

which, together with equation (A.9), implies

$$Q(\boldsymbol{\pi}) = \pm 1. \quad (\text{A.19})$$

(ii) Establishing $Q(\boldsymbol{\pi}) = Q(\boldsymbol{\pi}')$ whenever $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ are both odd or both even. In equation (A.11), let $\alpha_{\pi_i} = k(\delta_{\pi, \pi_i} + \delta_{\pi', \pi_i})$, where k is some constant. Then, using equations (A.18), (A.11) becomes

$$|Q(\boldsymbol{\pi}) + Q(\boldsymbol{\pi}')| = |1 + (-1)^\sigma|,$$

with $\sigma = 0$ or 1 , where $\boldsymbol{\pi}$ is any permutation and $\boldsymbol{\pi}'$ is of the form $\boldsymbol{\tau}\boldsymbol{\pi}$ where $\boldsymbol{\tau}$ is any transposition. Therefore

$$Q(\boldsymbol{\pi}') = (-1)^\sigma Q(\boldsymbol{\pi}). \quad (\text{A.20})$$

Now, let $\boldsymbol{\pi}'' = \boldsymbol{\tau}'\boldsymbol{\pi}'$, where $\boldsymbol{\tau}'$ is some transposition. Then, equation (A.20) implies that $Q(\boldsymbol{\pi}'') = (-1)^\sigma Q(\boldsymbol{\pi}')$. Combining this with $Q(\boldsymbol{\pi}') = (-1)^\sigma Q(\boldsymbol{\pi})$, we obtain

$$Q(\boldsymbol{\pi}'') = Q(\boldsymbol{\pi}). \quad (\text{A.21})$$

That is, any two permutations, $\boldsymbol{\pi}$, $\boldsymbol{\pi}''$, that are connected by a pair of transpositions have the same Q -value. But every even permutation can be written as a product of an even number of transpositions and the identity permutation. Therefore, all even permutations have the same Q -value, Q_e . Similarly, every odd permutation can be written as a product of an even number of transpositions and a given odd permutation. Therefore, all odd permutations have the same Q -value, Q_o . Finally, equation (A.20) implies that $Q_o = (-1)^\sigma Q_e$.

With these results for $Q(\boldsymbol{\pi})$, equation (A.18) becomes

$$H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}) = \pm \begin{cases} \sum_{\boldsymbol{\pi} \in S_N} (\text{sgn}(\boldsymbol{\pi}))^\sigma \alpha_{\boldsymbol{\pi}}, \\ \sum_{\boldsymbol{\pi} \in S_N} (\text{sgn}(\boldsymbol{\pi}))^\sigma \alpha_{\boldsymbol{\pi}}^*, \end{cases}$$

where $\text{sgn}(\boldsymbol{\pi})$ takes the value $+1$ or -1 according to whether $\boldsymbol{\pi}$ is even or odd.

Insofar as the probability of the transition of the system of N indistinguishable particles is concerned, the overall sign of H and the complex conjugation are irrelevant. More generally, consider a system, \mathcal{S} , that consists of subsystems \mathcal{S}_1 and \mathcal{S}_2 , where \mathcal{S}_1 consists of N indistinguishable particles of one type (say, electrons). Suppose, first, that \mathcal{S}_2 only contains particles that can be distinguished from those in \mathcal{S}_1 . As described above, let measurements L_1, \dots, L_N and M_1, \dots, M_N be performed on \mathcal{S}_1 at times t_1 and t_2 . Additionally, let measurements U and V be performed on \mathcal{S}_2 at t_1 and t_2 , respectively, yielding outcomes u and v . Let $\tilde{\alpha}_\pi$ be the amplitude of the transition of \mathcal{S} from $(\ell: u)$ to $(m: v)$ in which the particles in \mathcal{S}_1 are treated as distinguishable and make the transition described by $\boldsymbol{\pi}$. Then, by the same argument as described above, the amplitude of the process from $(\ell: u)$ to $(m: v)$ where the particles in \mathcal{S}_1 are treated as indistinguishable is given by $H(\tilde{\alpha}_{\pi_1}, \tilde{\alpha}_{\pi_2}, \dots, \tilde{\alpha}_{\pi_{N!}})$, and the transition probability is again unaffected by the overall sign or complex conjugation of H . In the case that \mathcal{S}_2 does contain, say, M particles that are indistinguishable from those in \mathcal{S}_1 , the boundaries of \mathcal{S}_1 must be redrawn to encompass them. The resulting situation, namely a system composed of subsystem \mathcal{S}'_1 , containing $N + M$ indistinguishable particles, and subsystem \mathcal{S}'_2 , containing only particles that are distinguishable from those in \mathcal{S}'_1 , is of the same type as one previously considered.

Therefore, in general, the overall sign of H and the complex conjugation are irrelevant insofar as predictions are concerned, and can be discarded without any loss of generality. Hence

$$H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_{N!}}) = \sum_{\boldsymbol{\pi} \in S_N} (\text{sgn}(\boldsymbol{\pi}))^\sigma \alpha_{\boldsymbol{\pi}}, \quad (\text{A.23})$$

where $\sigma = 0$ or $\sigma = 1$ is the only remaining degree of freedom, corresponding respectively to bosons and fermions.

The above result for the amplitude holds for a particular labelling of the outcomes at times t_1 and t_2 . However, the corresponding transition probability is invariant under relabelling of these outcomes. To see this, suppose that the outcomes at t_1 and t_2 are relabelled such that $\ell_i \rightarrow \ell_{\boldsymbol{\tau}'(i)}$ and $m_j \rightarrow m_{\boldsymbol{\tau}'(j)}$ for $i, j \in \{1, 2, \dots, N\}$,

where τ and τ' are each permutations of N elements. Then the transition $\pi_1 \rightarrow \pi$ is relabelled $\pi_1 \rightarrow \tau'^{-1}\pi_i\tau$. Now, $\text{sgn}(\pi) = \text{sgn}(\tau'^{-1}\pi_i\tau) = \text{sgn}(\tau'^{-1}\tau) \text{sgn}(\pi_i)$. Therefore, as a result of the outcome relabelling, either all even transitions go to even transitions, and odd transitions go to odd transitions; or all even transitions go to odd transitions, and all odd transitions go to even transitions. Now, note that equation (A.23) can be written as

$$H(\alpha_{\pi_1}, \alpha_{\pi_2}, \dots, \alpha_{\pi_N}) = \sum_{\pi \text{ even}} \alpha_{\pi} + \sigma \sum_{\pi \text{ odd}} \alpha_{\pi}. \quad (\text{A.24})$$

Thus, the effect of relabelling of outcomes is at most a change of sign of H . Hence, the corresponding transition amplitude is invariant under outcome relabelling, and there is no loss of generality in taking equation (A.23) for any particular labelling.

Appendix B. Solution of a pair of functional equations

We solve equations (14a) and (14b) under the condition that $z_1, z_2, z_1 + z_2 \in \overline{\mathbb{D}}$ with the aid of one of Cauchy's standard functional equations

$$h(x_1 + x_2) = h(x_1) + h(x_2), \quad (\text{B.1})$$

where h is a real function and $x_1, x_2 \in [-a, a]$ with $a \in \mathbb{R}$. Its continuous solution is $h(x) = \gamma x$ with $\gamma \in \mathbb{R}$ [48].

We first consider the solution of equation (14a) under the condition that z_1, z_2 lie in the closed half-unit circle, $\overline{\mathbb{D}}_{1/2}$, which automatically ensures that $z_1 + z_2$ lies in $\overline{\mathbb{D}}$. Setting $z_1 + z_2 = x + iy$, with $x, y \in [-1/2, 1/2]$, in equation (14a) gives

$$f(x + iy) = f(x) + f(iy).$$

Applying equation (14a) again on $f(x_1 + x_2)$ and $f(iy_1 + iy_2)$, with $x_1, x_2, y_1, y_2 \in [-1/2, 1/2]$ then implies

$$\begin{aligned} f(x_1 + x_2) &= f(x_1) + f(x_2) \\ f(iy_1 + iy_2) &= f(iy_1) + f(iy_2). \end{aligned}$$

The real and imaginary parts of both of these equations each have the form of equation (B.1), and therefore have solutions

$$f(x) = \alpha x \quad \text{and} \quad f(iy) = \beta y$$

with $\alpha, \beta \in \mathbb{C}$, so that, for all $(x + iy) \in \overline{\mathbb{D}}_{1/2}$

$$f(x + iy) = \alpha x + \beta y. \quad (\text{B.2})$$

Since this equation satisfies equation (14a) under the condition $z_1, z_2, z_1 + z_2 \in \overline{\mathbb{D}}$, it is the general solution under these conditions.

From equation (14b), considered under the condition $z_1, z_2, z_1 + z_2 \in \overline{\mathbb{D}}$

$$f(1 \cdot 1) = f(1)f(1) \quad \text{and} \quad f(i \cdot i) = f(i)f(i),$$

which, due to equation (B.2), imply that

$$\alpha = \alpha^2 \quad \text{and} \quad -\alpha = \beta^2.$$

These have solutions $(\alpha, \beta) = (0, 0), (1, i)$ and $(1, -i)$, which correspond to $f(z) = 0, f(z) = z$ and $f(z) = z^*$.

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